Stable Longitudinal Oscillations in Anisotropic Plasma. II*

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The dispersion equation for longitudinal oscillations in an infinite collisionless anisotropic plasma in a uniform magnetic field is investigated. It is shown that nonoscillating, exponentially growing modes cannot exist in a plasma whose equilibrium distribution function is a two-temperature Maxwellian. This is demonstrated by showing that the dispersion equation has no solution under these conditions. The Nyquist criterion for plasma instability is also invoked to prove that all longitudinal modes propagating along the magnetic field in such a plasma are stable.

IN a previous publication,[1] henceforth referred to as Part I, we had investigated the stability of longitudinal plasma oscillations in an infinite collisionless plasma in which the particles have anisotropic velocity distributions. More explicitly we had shown that purely growing (in time) modes with frequencies of the order of the ion cyclotron frequency and propagation at an angle to the uniform magnetic field cannot exist in a plasma in which the electrons are isotropic and the ions are anisotropic with a temperature ratio $T_\perp/T_{\parallel}>1$. Here $T_\perp$, and $T_{\parallel}$ refer to the ion temperatures perpendicular and parallel to the field, respectively. In this paper we shall remove these restrictions and demonstrate that anisotropic plasma is perfectly stable against purely growing longitudinal modes. Such nonoscillating and exponentially growing modes are found, for example, in the gravitational Rayleigh-Taylor instability.[2] This implies that unstable longitudinal oscillations in anisotropic plasma—if they exist—are necessarily of the "overstable"[3] kind. In addition, we shall demonstrate that all longitudinal modes traveling along the magnetic field are also stable.

We recall from Part I that we are considering an infinite collisionless plasma in a uniform magnetic field B_0 taken conveniently along the z direction. We assume that, in equilibrium configuration, the particles have a velocity distribution given by

$$f_0=\frac{1}{\pi^{3/2}\alpha_\perp^2\alpha_z}\exp\left[-\frac{v_\perp^2}{\alpha_\perp^2}-\frac{v_z^2}{\alpha_z^2}\right], \tag{1}$$

where $\alpha_\perp$ and α_z are the particle thermal velocities perpendicular and parallel to the field, respectively, and that there is no electric field E_0. We further assume that the system departs only slightly from equilibrium, and that all perturbations are of the form $\exp[i\mathbf{k}\cdot\mathbf{r}+i\omega t]$, where ω is the frequency and k is the wave number. In this case, the dispersion equation for longitudinal oscillations can be readily obtained from the linearized Vlasov equations and has the form[1]

$$1=\sum_j\frac{\omega_{pj}^2}{k^2}\sum_{n=-\infty}^{+\infty}2e^{-\lambda_j}I_n(\lambda_j)\times\left[-\frac{n\omega_{cj}}{k_z\alpha_{zj}\alpha_{\perp j}^2}Y(-\xi_j)-\frac{1}{\alpha_{zj}^2}+\frac{\xi_j}{\alpha_{zj}^2}Y(-\xi_j)\right], \tag{2}$$

where, as before, ω_p is the plasma frequency, ω_c is the cyclotron frequency, $\lambda=\frac{1}{2}(\gamma^2n_\perp^2)$, $\gamma=k\rho$, $n_\perp=k_\perp/k$, $n_z=k_z/k$, ρ is the particle radius of gyration, $k_\perp$ and k_z are the components of the wave vector perpendicular and parallel to the field, respectively, and the first summation is over the plasma species. The function $Y(-\xi)$ is the familiar dispersion function[4] usually defined by

$$Y(-\xi)=\int_{-\infty}^{+\infty}\frac{e^{-y^2}dy}{y+\xi}=i(\pi)^{1/2}e^{-\xi^2}\operatorname{erfc}(i\xi), \tag{3}$$

where

$$\operatorname{erfc}(i\xi)=\frac{2}{(\pi)^{1/2}}\int_{i\xi}^{\infty}e^{-y^2}dy,$$

$$\xi=\frac{\omega+n\omega_c}{k_z\alpha_z},$$

and $I_n(\lambda)=I_n(\lambda)$ is the Bessel function of the first kind of imaginary argument. We now depart from the analysis pursued in Part I, and simply utilize the second of Eqs. (3) along with known properties of the Bessel

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function $I_n(\lambda)$ to transform Eq. (2) into the form

$$1+k^2d_i^2+\frac{T_{\perp i}}{T_{\perp e}}$$
$$=\frac{T_{\perp i}}{T_{\perp e}}\int_0^\infty [i\omega_e+\tfrac{1}{2}(t_e^2-1)\gamma_e^2n_z^2x]$$
$$\times\exp\left[-i\omega_e x-\tfrac{1}{4}(\gamma_e^2t_e^2n_z^2)x^2-\gamma_e^2n_\perp^2\sin^2\left(\frac{x}{2}\right)\right]dx$$
$$+\int_0^\infty [i\omega_i+\tfrac{1}{2}(t_i^2-1)\gamma_i^2n_z^2x]$$
$$\times\exp\left[-i\omega_i x-\tfrac{1}{4}\gamma_i^2t_i^2n_z^2x^2-\gamma_i^2n_\perp^2\sin^2\left(\frac{x}{2}\right)\right]dx, \quad (4)$$

where the subscripts e and i denote electrons and ions, respectively. In Eq. (4) d_i is the Debye length defined in terms of the ion temperature, i.e., $d_i^2=KT_{\perp e}/4\pi Ne^2$, where K is the Boltzmann constant and N is the particle number density; $\omega_e=\omega/\omega_{ce}$, $\omega_i=\omega_i=\omega/\omega_{ci}$, $t_e^2=T_{ze}/T_{\perp e}$, and $t_i^2=T_{zi}/T_{\perp i}$. If we multiply Eq. (4) by $T_{\perp e}/T_{\perp i}$ and let $t_i^2=t_e^2=1$, we obtain the equation derived by Bernstein[5] for longitudinal oscillations in a plasma in which the electrons and the ions are isotropic but not in equilibrium with one another. We now consider the first integral in (4) and choose to put it in the form

$$Q_e=\int_0^\infty [i\omega_e+\tfrac{1}{2}(t_e^2-1)\gamma_e^2n_z^2x]$$
$$\times\exp[-i\omega_e x-\tfrac{1}{4}(t_e^2-1)\gamma_e^2n_z^2x^2]$$
$$\times\exp\left[-\gamma_e^2n_\perp^2\sin^2\left(\frac{x}{2}\right)-\tfrac{1}{4}\gamma_e^2n_z^2x^2\right]dx,$$

where we have made use of the fact that $n_z^2+n_\perp^2=1$. Assuming that $i\omega_e$ has a positive real part, and integrating by parts, we get

$$Q_e=1-\tfrac{1}{2}\gamma_e^2\int_0^\infty dx[n_\perp^2\sin x+n_z^2x]$$
$$\times\exp[-i\omega_e x-\tfrac{1}{4}\gamma_e^2n_z^2t_e^2x^2$$
$$-\tfrac{1}{2}(\gamma_e^2n_\perp^2+\gamma_e^2n_\perp^2\cos x)]. \quad (5)$$

A similar expression may be written for the integral containing the ion terms. With $(\gamma_e^2)T_{\perp i}/T_{\perp e}=(\gamma_i^2)M_e/M_i$, and letting $i\omega_e=i\beta_e+\nu_e$, $i\omega_i=i\beta_i+\nu_i$ with $\nu_e>0$ and $\nu_i>0$, we can finally write for Eq. (4),

$$-\frac{2k^2d_i^2}{\gamma_i^2}=\int_0^\infty dx[n_z^2x+n_\perp^2\sin x]$$
$$\times\left\{\frac{M_e}{M_i}\exp[-i\beta_e x-\nu_e x-\tfrac{1}{4}(\gamma_e^2t_e^2n_z^2x^2)\right.$$
$$-\tfrac{1}{2}(\gamma_e^2n_\perp^2+\gamma_e^2n_\perp^2\cos x)]$$
$$+\exp[-i\beta_i x-\nu_i x-\tfrac{1}{4}(\gamma_i^2t_i^2n_z^2x^2)$$
$$\left.-\tfrac{1}{2}(\gamma_i^2n_\perp^2+\gamma_i^2n_\perp^2\cos x)]\right\}. \quad (6)$$

For $\beta_e=\beta_i=0$, i.e., purely growing longitudinal modes, it is now possible to show that Eq. (6) has no solution. Consider first integrals of the type

$$Q=\int_0^\infty x\exp\left[-\nu x-\tfrac{1}{4}(\gamma^2t^2n_z^2x^2)-\gamma^2n_\perp^2\sin^2\left(\frac{x}{2}\right)\right]dx.$$

Since the limits of integration are from zero to infinity, it is readily seen that such integrals are positive definite. Consider next the second type of integrals in Eq. (6), namely

$$\int_0^\infty dx\sin x$$
$$\times\exp[-\nu x-\tfrac{1}{4}(\gamma^2t^2n_z^2x^2)-\tfrac{1}{2}(\gamma^2n_\perp^2+\gamma^2n_\perp^2\cos x)]$$
$$\equiv\int_0^\infty dx\exp[-\nu x-\tfrac{1}{4}(\gamma^2t^2n_z^2x^2)-\lambda]\sin x e^{\lambda\cos x}. \quad (7)$$

The periodic functions $\sin x$ and $e^{\cos x}$ which appear in the integrand of Eq. (7) have the same periodicity. Moreover, in the interval from $x=0$ to $x=2\pi$ the product $\sin x e^{\lambda\cos x}$ is antisymmetric about $x=\pi$, and the area under such a curve is identically zero. In Eq. (7), however, this product is multiplied by an exponentially decreasing function of x which makes the area in the first half of the cycle—the positive area—larger than that in the second half—the negative area. Since the integral from $x=0$ to $x=\infty$ can be viewed as sum of integrals of the limits, it is clear that such an integral has a positive definite value. In view of this and the previous argument, the right-hand side of the dispersion Eq. (6) is always positive definite. Since the left-hand side is negative definite, it is evident that the equation has no solutions with ν (or ω)>0 and hence no purely growing modes.

The results of the above section are particularly useful in the investigation of the stability of all longitudinal modes which propagate along the magnetic field. We utilize these results in connection with the Nyquist criterion[6] which we now employ to ascertain

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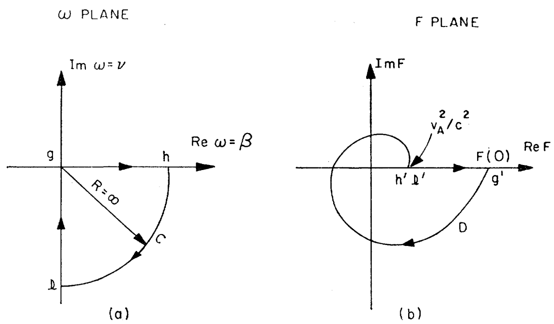

FIG. 1. Nyquist diagram.

the stability of these modes. We find it convenient first to rewrite the dispersion Eq. (6) in the form

$$F(\omega)=\frac{v_A^2}{c^2}+\int_0^\infty dx\left\{\frac{M_e}{M_i}\exp\left[-i\omega_e x-\tfrac{1}{4}(\gamma_e^2t_e^2n_z^2x^2)-\gamma_e^2n_\perp^2\sin^2\left(\frac{x}{2}\right)\right]+\exp\left[-i\omega_i x-\tfrac{1}{4}(\gamma_i^2t_i^2n_z^2x^2)-\gamma_i^2n_\perp^2\sin^2\left(\frac{x}{2}\right)\right]\right\}\{n_z^2x+n_\perp^2\sin x\}=0, \quad (8)$$

where we have replaced $2k^2d_i/\gamma^2$ by its equivalent v_A^2/c^2, and where v_A is the Alfvén speed and c is the velocity of light. We observe that $F(\omega)$ is analytic in the lower half of the complex ω plane. We further note that the zeros of $F(\omega)$ occur in pairs which are mirror images of one another, i.e., if ω_r is a root then $(i\omega_r)^*$, where * denotes the complex conjugate, is also a root. In view of this we need to examine only one quadrant in the lower half of the ω plane for instability. Consider therefore the closed contour C which encompasses the whole fourth quadrant shown in Fig. 1(a). This curve can then be mapped into a closed curve such as D in the complex F plane shown in Fig. 1(b). As a consequence of Cauchy's theorem the number of zeros of $F(\omega)$ inside the contour C is equal to the number of times the curve D encloses the origin. We have already demonstrated that $F(\omega)$ is positive definite for all ω's which are negative imaginary, i.e., along line gl in Fig. 1(a). This is equivalent to saying that the line gl in the ω plane maps into the line $g'l'$ in the F plane.

Moreover, it is a simple matter to show that $F(\omega)$ tends toward v_A^2/c^2 for ω's along the arc hl as its radius tends to infinity. In view of this we need to examine $F(\omega)$ for real ω's only, i.e., for all β's from zero to infinity. To do this, we return to Eq. (8) and replace $i\omega_e$ and $i\omega_i$ by $i\beta_e$ and $i\beta_i$, respectively, and set $n_\perp=0$ (propagation along the field). It becomes

$$F(\omega)=\frac{v_A^2}{c^2}+\int_0^\infty\left\{\frac{M_e}{M_i}\exp[-i\beta_e x-\tfrac{1}{4}(\gamma_e^2t_e^2x^2)]+\exp[-i\beta_i x-\tfrac{1}{4}(\gamma_i^2t_i^2x^2)]\right\}xdx. \quad (9)$$

Performing the integrations and rearranging we obtain

$$F(\omega)=\frac{v_A^2}{c^2}+\frac{2M_e}{M_i\gamma_e^2t_e^2}+\frac{2}{\gamma_i^2t_i^2}+\frac{2M_e\beta_e}{M_i\gamma_e^3t_e^3}Y\left(\frac{\beta_e}{\gamma_e t_e}\right)-i\frac{4(\pi)^{1/2}M_e\beta_e}{M_i\gamma_e^3t_e^3}\exp\left[-\frac{\beta_e^2}{\gamma_e^2t_e^2}\right]-\frac{2\beta_i}{\gamma_i^3t_i^3}Y\left(\frac{\beta_i}{\gamma_i t_i}\right)-i\frac{4(\pi)^{1/2}\beta_i}{\gamma_i^3t_i^3}\exp\left[-\frac{\beta_i^2}{\gamma_i^2t_i^2}\right], \quad (10)$$

where $Y(\beta|\gamma t)$ is the plasma dispersion function defined earlier. At this point it is sufficient to examine only the imaginary part of (10) to determine whether the closed curve in the F plane encloses the origin. We recall that for a real argument—such as we have here—the imaginary part of the dispersion function Y is given by[4]

$$\mathrm{Im}Y\left(\frac{\beta}{\gamma t}\right)=i(\sqrt{\pi})^{\frac{1}{2}}\exp\left[-\frac{\beta^2}{\gamma^2t^2}\right] \quad (11)$$

so that the imaginary part of $F(\omega)$ can be written as

$$\mathrm{Im}F(\omega)=-\frac{2(\pi)^{1/2}M_e\beta_e}{M_i\gamma_e^3t_e^3}\exp\left[-\frac{\beta_e^2}{\gamma_e^2t_e^2}\right]-\frac{2(\pi)^{1/2}\beta_i}{\gamma_i^3t_i^3}\exp\left[-\frac{\beta_i^2}{\gamma_i^2t_i^2}\right]. \quad (12)$$

It is clear that the $\mathrm{Im}F(\omega)$ is negative for all positive β's and for all values of the parameters, γ, t, M_e/M_i which are always positive. $F(\omega)$ cannot, therefore, encircle the origin since this requires one or more changes in the sign of its imaginary part. Thus roots in the lower half of the ω plane do not exist and longitidinal oscillations propagating along the magnetic field are completely stable.

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